HOMEWORK 10
MATH 430, SPRING 2014

Problem 1. *Recall that group 3 of the logical axioms is generalizations of formulas of the kind* $\forall x(\alpha \to \beta) \to (\forall x\alpha \to \forall x\beta)$. *Write down a* Δ_1 *formula* $\phi(v)$*, such that* $\mathfrak{A} \models \phi[e]$ *iff* e *codes a formula in group 3 of the logical axioms. I.e.* ϕ_e *is of the form* $\forall x_1, ... \forall x_n(\forall x(\alpha \to \beta) \to (\forall x\alpha \to \forall x\beta))$.

Problem 2. *Write down a* Δ_1 *formula* $\phi(v)$*, such that* $\mathfrak{A} \models \phi[e]$ *iff* e *codes a sentence (i.e. a formula with no free variables).*

For $i < n$, define the projection $P_i^n : \mathbb{N}^n \to \mathbb{N}$ to be $P_i^n(x_0, ..., x_{n-1}) = x_i$. The **primitive recursive functions** are functions $f : \mathbb{N}^k \to \mathbb{N}$ that are build up from the constant function $x \mapsto 0$, successor, projection, by applying composition, and the primitive recursive operation:

- $f(0, x_1, ..., x_{k-1}) = g(x_1, ..., x_{k-1})$,
- $f(S(y), x_1, ..., x_n) = h(y, f(y, x_1, ..., x_{k-1}), x_1, ..., x_{k-1})$

where g and f are primitive recursive.

Problem 3. *Show that addition is primitive recursive. I.e. you have to show that the function* $(x, y) \mapsto x + y$ *can be written as above.*

For a function $f : \mathbb{N}^k \to \mathbb{N}$, ϕ_f will denote the formula such that

$$\mathfrak{A} \models \phi_f[a_0, ..., a_{k-1}, b] \text{ iff } f(a_0, ..., a_{k-1}) = b.$$

For example, if f is the addition function, then $\phi_f(x, y, z)$ is the formula $x + y = z$. Note that for each of the projection, successor, addition, multiplication, the corresponding formula is atomic, and therefore Δ_0.

Problem 4. *Suppose that* f *is defined by primitive recursion from functions* g, h *as above and suppose that* ϕ_g, ϕ_h *are* Δ_1. *Write down* ϕ_f *in terms of* ϕ_g, ϕ_h *and show it is also* Δ_1.

Problem 5. *Suppose that* $f : \mathbb{N}^k \to \mathbb{N}$ *and that* ϕ_f *is* Σ_1. *Show that* ϕ_f *is also equivalent to a* Π_1 *formula, and therefore* Δ_1.
Hint: It is enough to show that $\neg\phi_f$ *is equivalent to* Σ_1. *Then use that negations of* Σ_1 *formulas are equivalent to* Π_1 *formulas.*

FYI: The **partial recursive functions** are build up from the primitive recursive functions by including "minimization": if g is recursive then

$$f(x_0, ...x_{n-1}) := \text{the least } y \text{ such that } g(y, x_0, ...x_{n-1}) = 0$$

is partial recursive i.e. its domain is a subset of $\mathbb{N}$, but it may not always be defined. f is partial recursive iff ϕ_f is Σ_1; f is total recursive iff ϕ_f is Δ_1.